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A PRODUCT PROPERTY FOR THE PLURICOMPLEX ENERGY

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Abstract

In this note we prove a product property for the pluricomplex energy, and then give some applications.

1. Introduction

Throughout this note assume that $\Omega \subseteq \mathbb{C}^n$, $n \geq 1$, is hyperconvex set. Recall that an open set $\Omega \subseteq \mathbb{C}^n$ is called hyperconvex if it is bounded, connected, and if there exists a bounded plurisubharmonic function $\varphi\colon \Omega \to (-\infty, 0)$ such that the closure of the set $\{z \in \Omega\colon \varphi(z) < c\}$ is compact in Ω, for every $c \in (-\infty, 0)$. The family of all bounded plurisubharmonic functions φ defined on Ω such that $\lim_{z\to\xi} \varphi(z) = 0$, for every $\xi \in \partial\Omega$, and $\int_\Omega (dd^c\varphi)^n < +\infty$, is denoted by $\mathcal{E}_0(\Omega)$. The family $\mathcal{E}_0$ is the analog of potentials for subharmonic functions in the classical potential theory. Here $(dd^c\,\cdot\,)^n$ is the complex Monge–Ampère operator. The aim of this note is to prove the following theorem.

Main Theorem. *Assume that $\Omega_1 \subset \mathbb{C}^{n_1}$, $n_1 \geq 1$, and $\Omega_2 \subset \mathbb{C}^{n_2}$, $n_2 \geq 1$, are two bounded hyperconvex domains, and let $u_1 \in \mathcal{E}_0(\Omega_1)$, $u_2 \in \mathcal{E}_0(\Omega_2)$. If $u(z_1, z_2) = \max(u_1(z_1), u_2(z_2))$, then*

$$\int_{\Omega_1\times\Omega_2} h(u)(dd^c u)^{n_1+n_2} = \int_{\Omega_1\times\Omega_2} h(u)(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}, \tag{1.1}$$

for all upper semicontinuous functions $h\colon (-\infty, 0] \to \mathbb{R}$.

It should be noted that the integrals in equality (1.1) can be, at the same time, $-\infty$. A sufficient condition to make sure that they are finite is to additional assume that h is bounded. Equality (1.1) is also valid for all decreasing functions $h\colon (-\infty, 0) \to [0, +\infty)$ (Corollary 2.2).

In the rest of this note we give some applications of our main theorem. Now we follow [6], and define $\mathcal{E}_p(\Omega)$, $p > 0$, to be the class of plurisubharmonic functions u defined on Ω for which there exists a decreasing sequence $[u_j]$, $u_j \in \mathcal{E}_0$, that converges

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pointwise to u on Ω, as j tends to $+\infty$, and

$$\sup_{j\geq 1}\int_{\Omega}(-u_j)^p(dd^c u_j)^n = \sup_{j\geq 1} e_p(u_j) < +\infty.$$

If $u \in \mathcal{E}_p(\Omega)$, then $e_p(u) < +\infty$ ([6, 10]). It should be noted that it follows from [6] that any function in $\mathcal{E}_p$ is in $\mathcal{E}$ and hence by [7] the operator $(dd^c\cdot)^n$ is well defined on $\mathcal{E}_p$, $p > 0$. The class $\mathcal{E}$ is the largest set of non-positive plurisubharmonic functions Ω for which the complex Monge–Ampère operator is well-defined ([7]). These convex cones are useful outside the field of pluripotential theory (see e.g. [2, 12]). If $u_1 \in \mathcal{E}_{p_1}(\Omega_1)$, $u_2 \in \mathcal{E}_{p_2}(\Omega_2)$, and $u(z_1, z_2) = \max(u_1(z_1), u_2(z_2))$, then we prove that $u \in \mathcal{E}_{p_1+p_2}(\Omega_1 \times \Omega_2)$, and

$$e_{p_1+p_2}(u) \leq e_{p_1}(u_1)e_{p_2}(u_2)$$

(Corollary 3.1). By using the idea from Example 2.6 in [3] we construct an example that shows that Corollary 3.1 is optimal in the following sense: Let $p_1, p_2 > 0$, then there exist functions $u_1 \in \mathcal{E}_{p_1}(\Omega_1)$, $u_2 \in \mathcal{E}_{p_2}(\Omega_2)$ such that

$$u(z_1, z_2) = \max(u_1(z_1), u_2(z_2)) \notin \bigcup_{q\geq 0/q\neq p_1+p_2} \mathcal{E}_q(\Omega_1 \times \Omega_2)$$

(Example 3.3). Furthermore, our main theorem yields, in Corollary 2.1, Wiklund's product property for $\mathcal{F}$. This result was first obtained by Wiklund in [17].

Before proceeding, let us introduce some convenient notations. Let $u \in \mathcal{E}$, then by Theorem 5.11 in [7] there exist functions $\phi_u \in \mathcal{E}_0$ and $f_u \in L^1_{\text{loc}}((dd^c\phi_u)^n)$, $f_u \geq 0$ such that $(dd^c u)^n = f_u(dd^c\phi_u)^n + \beta_u$. The non-negative measure β_u is such that there exists a pluripolar set $A \subseteq \Omega$ such that $\beta_u(\Omega \setminus A) = 0$. We shall use the notation that $\alpha_u = f_u(dd^c\phi_u)^n$ and β_u refereing to the decomposition discussed here. If $u_1 \in \mathcal{E}(\Omega_1)$, $u_2 \in \mathcal{E}(\Omega_2)$, then we prove that $\max(u_1, u_2) \in \mathcal{E}(\Omega_1 \times \Omega_2)$, and $\beta_{\max(u_1,u_2)} = \beta_{u_1} \otimes \beta_{u_2}$ (Corollary 2.1 and Theorem 4.5).

For further information about pluripotential theory, and the complex Monge–Ampère operator, we refer to the monographs by Klimek ([14]), and Kołodziej ([15]).

2. Proof of Main Theorem

Proof of Main Theorem. Set $\Omega = \Omega_1 \times \Omega_2$, $n = n_1 + n_2$. Without loss of generality we can assume that $u_1, u_2 < 0$.

CASE I: Assume that $u_1 \in \mathcal{E}_0(\Omega_1) \cap C^\infty(\Omega_1)$, $u_2 \in \mathcal{E}_0(\Omega_2) \cap C^\infty(\Omega_2)$, and $h \in C_0^\infty((-\infty, 0), \mathbb{R})$. To see that $h(u)$ is the difference of two functions in $\mathcal{E}_0(\Omega)$ we show that there are two convex and increasing functions $h_1, h_2 \in C((-\infty, 0), \mathbb{R})$ with $h_1(0) =$

$h_2(0) = 0$ and $h_1 + h_2 \geq Mx$ for a constant $M > 0$. Explicitly, choose $a < 0$ and $b > 0$ such that

$$a < \inf_{\operatorname{supp} h}(h(x) + Se^x - b) \leq \sup_{x<0}(h(x) + Se^x - b) \leq 0,$$

where $S > 0$ is so large that $h(x) + Se^x$ is convex and increasing. Now choose $M > 0$ such that $Mx < a$ on $\operatorname{supp} h$. Then set

$$h_1(x) = \max(h(x) + Se^x - b,\, Mx) \quad \text{and} \quad h_2(x) = \max(Se^x - b,\, Mx).$$

Assume for the moment that $u \in \mathcal{E}_0(\Omega_1 \times \Omega_2)$ (this is later proved in Case V). The facts that $u = u_1$ on the support of $(dd^c u)^{n_2} \wedge dd^c h(u)$, and $u = u_2$ on the support of $dd^c h(u) \wedge (dd^c u_1)^{n_1}$, yield together with integration by parts ([7]) that

$$\begin{aligned}
\int_\Omega h(u)(dd^c u)^n &= \int_\Omega u(dd^c u)^{n-1} \wedge dd^c h(u) = \int_\Omega u_1(dd^c u)^{n-1} \wedge dd^c h(u) \\
&= \int_\Omega h(u)(dd^c u)^{n-1} \wedge dd^c u_1 = \cdots = \int_\Omega h(u)(dd^c u)^{n_2} \wedge (dd^c u_1)^{n_1} \\
&= \int_\Omega u(dd^c u)^{n_2-1} \wedge dd^c h(u) \wedge (dd^c u_1)^{n_1} \\
&= \int_\Omega u_2(dd^c u)^{n_2-1} \wedge dd^c h(u) \wedge (dd^c u_1)^{n_1} \\
&= \int_\Omega h(u)(dd^c u)^{n_2-1} \wedge (dd^c u_1)^{n_1} \wedge dd^c u_2 \\
&= \cdots \\
&= \int_\Omega h(u)(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}.
\end{aligned}$$

Thus,

$$\int_\Omega h(u)(dd^c u)^n = \int_\Omega h(u)(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}.$$

CASE II: Assume that $u_1 \in \mathcal{E}_0(\Omega_1)$, $u_2 \in \mathcal{E}_0(\Omega_2)$, and $h \in C_0^\infty((-\infty, 0), \mathbb{R})$. From [8] it follows that there exist two decreasing sequences $[u_1^j]$, $u_1^j \in \mathcal{E}_0(\Omega_1) \cap C^\infty(\Omega_1)$, and $[u_2^j]$, $u_2^j \in \mathcal{E}_0(\Omega_2) \cap C^\infty(\Omega_2)$, that converge pointwise to u_1 and u_2, respectively, as $j \to +\infty$. Set $u^j = \max(u_1^j, u_2^j)$. Case I yields that

$$\begin{aligned}
\int_\Omega (h_1(u^j) - h_2(u^j))(dd^c u^j)^n &= \int_\Omega h(u^j)(dd^c u^j)^n \\
&= \int_\Omega h(u^j)(dd^c u_1^j)^{n_1} \wedge (dd^c u_2^j)^{n_2} \\
&= \int_\Omega (h_1(u^j) - h_2(u^j))(dd^c u_1^j)^{n_1} \wedge (dd^c u_2^j)^{n_2}.
\end{aligned}$$

If we let $j \to +\infty$, then Proposition 5.1 [7] shows that the left hand side tends to $\int_\Omega h(u)(dd^c u)^n$, and also using Fubini's theorem we see that the right hand tends to $\int_\Omega h(u)(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}$.

CASE III: For this and the next case assume that $h \in C((-\infty, 0], \mathbb{R})$ and let

$$M = \sup\{|u_1(z_1)| + |u_2(z_2)| : z_1 \in \Omega_1,\ z_2 \in \Omega_2\}.$$

Furthermore, we choose a sequence $[h_j]$, $h_j \in C_0^\infty((-\infty, 0), \mathbb{R})$ such that

$$\sup_{j \geq 1} \sup\{|h_j(t)| : t \in [-M, 0]\} < +\infty,$$

and which converges uniformly to h for all compact sets of $[-M, 0)$ as $j \to +\infty$. From Case II we now get that

$$\int_\Omega h_j(u)(dd^c u)^n = \int_\Omega h_j(u)(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}. \tag{2.1}$$

This case is finished by letting $j \to +\infty$ and using Lebesgue's dominated convergence theorem together with (2.1).

CASE IV: In general case, we choose a decreasing sequence $[h_j]$, h_j: $C((-\infty, 0], \mathbb{R})$, that converges pointwise to h on $[-M, 0]$ as $j \to +\infty$. By Case III we have that

$$\int_\Omega h_j(u)(dd^c u)^n = \int_\Omega h_j(u)(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2},$$

and this proof can be finished as Case III.

CASE V: It remains to show that $u = \max(u_1, u_2) \in \mathcal{E}_0(\Omega)$. This follows immediately from [17], but here we give a direct proof. Fix $z_0 \in \Omega_1$, and $w_0 \in \Omega_2$. Let g_1, and g_2, be the pluricomplex Green functions defined on Ω_1, and Ω_2, with poles in z_0, and w_0, respectively. It follows from [11] and Proposition 3.4 in [19] that $\max(g_1, g_2, -1) \in \mathcal{E}_0(\Omega)$. Define

$$u_1^j = \max(u_1,\ j\max(g_1, -1)),\quad u_2^j = \max(u_2,\ j\max(g_2, -1)),\quad \text{and}\quad u^j = \max(u_1^j, u_2^j).$$

Then $\max(u_1^j, u_2^j) \geq j\max(g_1, g_2, -1) \in \mathcal{E}_0(\Omega)$ and we have proved in Case III that

$$\int_{\Omega_1\times\Omega_2} (dd^c u^j)^{n_1+n_2} = \int_{\Omega_1} (dd^c u_1^j)^{n_1} \int_{\Omega_2} (dd^c u_2^j)^{n_2} \leq \int_{\Omega_1} (dd^c u_1)^{n_1} \int_{\Omega_2} (dd^c u_2)^{n_2},$$

and since $[u^j]$ decreases pointwise to u as $j \to +\infty$, it follows that $u \in \mathcal{E}_0(\Omega)$. □

In Corollary 2.1, we show how our main theorem yields Wiklund's product property for $\mathcal{F}$. The result in Corollary 2.1 was first obtained in [17].

Corollary 2.1. *Assume that* $\Omega_1 \subset \mathbb{C}^{n_1}$, $n_1 \geq 1$, *and* $\Omega_2 \subset \mathbb{C}^{n_2}$, $n_2 \geq 1$, *are two bounded hyperconvex domains, and let* $u_1 \in \mathcal{F}(\Omega_1)$, $u_2 \in \mathcal{F}(\Omega_2)$. *If* $u(z_1, z_2) = \max(u_1(z_1), u_2(z_2))$, *then* $u \in \mathcal{F}(\Omega_1 \times \Omega_2)$, *and*

$$\int_{\Omega_1\times\Omega_2} (dd^c u)^{n_1+n_2} = \int_{\Omega_1} (dd^c u_1)^{n_1} \int_{\Omega_2} (dd^c u_2)^{n_2}.$$

Furthermore, if $u_1 \in \mathcal{E}(\Omega_1)$, $u_2 \in \mathcal{E}(\Omega_2)$ *then* $u(z_1, z_2) = \max(u_1(z_1), u_2(z_2)) \in \mathcal{E}(\Omega_1 \times \Omega_2)$.

Proof. We set $\Omega = \Omega_1 \times \Omega_2$ and $n = n_1 + n_2$. From [8] it follows that there exist two decreasing sequences $[u_1^j]$, $u_1^j \in \mathcal{E}_0(\Omega_1) \cap C^\infty(\Omega_1)$, and $[u_2^j]$, $u_2^j \in \mathcal{E}_0(\Omega_2) \cap C^\infty(\Omega_2)$, that converge pointwise to u_1 and u_2, respectively, as $j \to +\infty$. An application of the main theorem gives the first two statements. The third statement now follows from the second, since every function in $\mathcal{E}$ is locally equal to a function in $\mathcal{F}$. $\square$

Corollary 2.2. *Assume that* $\Omega_1 \subset \mathbb{C}^{n_1}$, $n_1 \geq 1$, *and* $\Omega_2 \subset \mathbb{C}^{n_2}$, $n_2 \geq 1$, *are two bounded hyperconvex domains, and let* $u_1 \in \mathcal{E}_0(\Omega_1)$, $u_2 \in \mathcal{E}_0(\Omega_2)$. *If* $u(z_1, z_2) = \max(u_1(z_1), u_2(z_2))$, *then*

$$\int_{\Omega_1\times\Omega_2} h(u)(dd^c u)^{n_1+n_2} = \int_{\Omega_1\times\Omega_2} h(u)(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2},$$

for all decreasing functions $h\colon (-\infty, 0) \to [0, +\infty)$.

Proof. Let $\Omega = \Omega_1 \times \Omega_2$, $n = n_1 + n_2$, and

$$M = \sup\{|u_1(z_1)| + |u_2(z_2)| \colon z_1 \in \Omega_1,\ z_2 \in \Omega_2\}.$$

Let $[h_j]$, $h_j\colon C((-\infty, 0], \mathbb{R})$, be a sequence that converges pointwise to h, as $j \to +\infty$, and

$$\sup_{j\geq 1} \sup\{|h_j(t)| \colon t \in [-M, 0)\} < +\infty.$$

By our main theorem we have that

$$\int_{\Omega} h_j(u)(dd^c u)^n = \int_{\Omega} h_j(u)(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}.$$

Let $j \to +\infty$, then Lebesgue's dominated convergence theorem completes this proof. $\square$

3. Some applications

Corollary 3.1. *Assume that* $\Omega_1 \subset \mathbb{C}^{n_1}$, $n_1 \geq 1$, *and* $\Omega_2 \subset \mathbb{C}^{n_2}$, $n_2 \geq 1$, *are two bounded hyperconvex domains, and let* $u_1 \in \mathcal{E}_{p_1}(\Omega_1)$, $u_2 \in \mathcal{E}_{p_2}(\Omega_2)$. *If* $u(z_1, z_2) = \max(u_1(z_1), u_2(z_2))$, *then* $u \in \mathcal{E}_{p_1+p_2}(\Omega_1 \times \Omega_2)$, *and*

$$e_{p_1+p_2}(u) \leq e_{p_1}(u_1)e_{p_2}(u_2).$$

Proof. Set $\Omega = \Omega_1 \times \Omega_2$, $n = n_1 + n_2$ and $p = p_1 + p_2$. By Lemma 2.1 in [10] we can find two decreases sequences $[u_1^j]$, $u_1^j \in \mathcal{E}_0(\Omega_1)$, and $[u_2^j]$, $u_2^j \in \mathcal{E}_0(\Omega_2)$, that converge pointwise to u_1 and u_2, respectively, as $j \to +\infty$. Furthermore, we have that $[(dd^c u_1^j)^{n_1}]$ and $[(dd^c u_2^j)^{n_2}]$ are increasing sequences that converge weakly to $(dd^c u_1)^{n_1}$ and $(dd^c u_2)^{n_2}$, as $j \to +\infty$. Let $[u^j]$ be the decreasing sequence that is defined by $u^j = \max(u_1^j, u_2^j) \in \mathcal{E}_0(\Omega)$. This construction yields that $[u^j]$ converges pointwise to $u = \max(u_1, u_2)$. Using the main theorem with $h(t) = (-t)^p$, and Fubini's theorem we have that

$$\begin{aligned}
e_p(u) &\leq \varliminf_{j\to+\infty} e_p(u^j) = \varliminf_{j\to+\infty} \int_\Omega h(u^j)(dd^c u^j)^n \\
&= \varliminf_{j\to+\infty} \int_\Omega h(u^j)(dd^c u_1^j)^{n_1} \wedge (dd^c u_2^j)^{n_2} \\
&\leq \varliminf_{j\to+\infty} \int_\Omega (-u_1^j)^{p_1}(-u_2^j)^{p_2}(dd^c u_1^j)^{n_1} \wedge (dd^c u_2^j)^{n_2} \leq \varliminf_{j\to\infty} e_{p_1}(u_1^j)e_{p_2}(u_2^j) \\
&= e_{p_1}(u_1)e_{p_2}(u_2).
\end{aligned}$$

□

We will need the following lemma in Example 3.3.

Lemma 3.2. *Let* $0 \leq p \leq q$. *Then*

$$\mathcal{E}_p(\Omega) \cap \mathcal{E}_q(\Omega) \subset \mathcal{E}_t(\Omega) \quad \textit{for all} \quad p \leq t \leq q.$$

Proof. For $0 \leq p \leq q$ choose $0 \leq \alpha \leq 1$ such that $t = \alpha p + (1-\alpha)q$. By Hölder's inequality we have that for each $v \in \mathcal{E}_0(\Omega)$ it holds that

$$\begin{aligned}
\int_\Omega (-v)^t (dd^c v)^n &= \int_\Omega (-v)^{\alpha p+(1-\alpha)q}(dd^c v)^n \\
&\leq \left(\int_\Omega (-v)^p (dd^c v)^n\right)^\alpha \left(\int_\Omega (-v)^q (dd^c v)^n\right)^{1-\alpha}.
\end{aligned}$$

Hence,

$$e_t(v) \leq e_p(v)^\alpha e_q(v)^{1-\alpha}. \tag{3.1}$$

Now let $u \in \mathcal{E}_p(\Omega) \cap \mathcal{E}_q(\Omega)$. Lemma 2.1 in [10] implies that there exists a decreasing sequence $[u_j]$, $u_j \in \mathcal{E}_0$, that converges pointwise to u as $j \to +\infty$,

$$\lim_{j\to+\infty} e_p(u_j) = e_p(u), \quad \text{and} \quad \lim_{j\to+\infty} e_q(u_j) = e_q(u).$$

Inequality (3.1) yields that

$$\sup_j e_t(u_j) \le \sup_j e_p(u_j)^\alpha e_q(u_j)^{1-\alpha} \le e_p(u)^\alpha e_q(u)^{1-\alpha}.$$

Thus, $u \in \mathcal{E}_t$ with $e_t(u) \le e_p(u)^\alpha e_q(u)^{1-\alpha}$. □

EXAMPLE 3.3. Assume that $\Omega_1 \subset \mathbb{C}^{n_1}$, $n_1 \ge 1$, and $\Omega_2 \subset \mathbb{C}^{n_2}$, $n_2 \ge 1$, are two bounded hyperconvex domains. In this example we show that there exist functions $u_1 \in \mathcal{E}_{p_1}(\Omega_1)$, and $u_2 \in \mathcal{E}_{p_2}(\Omega_2)$ such that

$$u(z_1, z_2) = \max(u_1(z_1), u_2(z_2)) \notin \bigcup_{q\ge 0/q\ne p_1+p_2} \mathcal{E}_q(\Omega_1 \times \Omega_2).$$

PART I: In this part we prove that for given $q > 0$ with $q \ne p_1 + p_2$, there exist functions $u_1 \in \mathcal{E}_{p_1}(\Omega_1)$, $u_2 \in \mathcal{E}_{p_2}(\Omega_2)$ such that $u(z_1, z_2) = \max(u_1(z_1), u_2(z_2)) \notin \mathcal{E}_q(\Omega_1 \times \Omega_2)$. Let $g_1(z_1) = g_{\Omega_1}(z_1, a_1)$, and $g_2(z_2) = g_{\Omega_2}(z_2, a_2)$ be the pluricomplex Green function defined on Ω_k with pole at $a_k \in \Omega_k$, $k = 1, 2$. Let also $p_1, p_2 > 0$.

CASE I: Assume that $q > p_1 + p_2$, and let $q_1 > p_1$, $q_2 > p_2$ be such that $q = q_1 + q_2$. For each $j \in \mathbb{N}$ set

$$v_1^j = \max(j^{-q_1/n_1} g_1, -j), \quad v_2^j = \max(j^{-q_2/n_2} g_2, -j), \quad \text{and} \quad v^j = \max(v_1^j, v_2^j).$$

We have that

$$\lim_{j\to+\infty} e_{p_1}(v_1^j) = \lim_{j\to+\infty} (2\pi)^{n_1} j^{p_1-q_1} = 0,$$

and

$$\lim_{j\to+\infty} e_{p_2}(v_2^j) = \lim_{j\to+\infty} (2\pi)^{n_2} j^{p_2-q_2} = 0.$$

Therefore by Lemma 2.5 in [3] we can choose subsequences of $[v_1^j]$, $[v_2^j]$, to get that

$$u_1 = \left(\sum_{j=1}^{+\infty} v_1^j\right) \in \mathcal{E}_{p_1}(\Omega_1), \quad \text{and} \quad u_2 = \left(\sum_{j=1}^{+\infty} v_2^j\right) \in \mathcal{E}_{p_2}(\Omega_2). \tag{3.2}$$

Since there is no risk of ambiguity we also call these subsequences $[v_1^j]$, $[v_2^j]$. Corollary 2.1, and Lemma 4.1 imply that $e_q(v^j) = (2\pi)^{n_1+n_2}$. Hence,

$$e_q\left(\sum_{j=1}^{k} v^j\right) \geq \sum_{j=1}^{k} e_q(v^j) = (2\pi)^{n_1+n_2} k.$$

Thus, $\sum_{j=1}^{+\infty} v^j \notin \mathcal{E}_q(\Omega_1 \times \Omega_2)$. On the other hand, we have for u_1, u_2 defined in (3.2) that $u = u(z_1, z_2) = \max(u_1(z_1), u_2(z_2)) \leq \sum_{j=1}^{+\infty} v^j$, which implies that $u \notin \mathcal{E}_q(\Omega_1 \times \Omega_2)$.

CASE II: Assume that $q < p_1 + p_2$, and let $q_1 < p_1$, $q_2 < p_2$ be such that $q = q_1 + q_2$. For each $j \in \mathbb{N}$ set

$$v_1^j = \max\left(j^{q_1/n_1} g_1, -\frac{1}{j}\right), \quad v_2^j = \max\left(j^{q_2/n_2} g_2, -\frac{1}{j}\right), \quad \text{and} \quad v^j = \max(v_1^j, v_2^j).$$

Then it is proved in a similar manner as in Case I that

$$u = u(z_1, z_2) = \max(u_1(z_1), u_2(z_2)) \notin \mathcal{E}_q(\Omega_1 \times \Omega_2).$$

PART II: By using Part I we shall complete this example. Set $q_j = p + (-1)^j/j$. For each $j \in \mathbb{N}$ Part I ensures the existence of functions $u_1^j \in \mathcal{E}_{p_1}(\Omega_1)$, $u_2^j \in \mathcal{E}_{p_2}(\Omega_2)$, with

$$u^j = \max(u_1^j, u_2^j) \notin \mathcal{E}_{q_j}(\Omega_1 \times \Omega_2).$$

Choose a positive sequence $\{\varepsilon_j\}$ of real numbers such that

$$u_1 = \left(\sum_{j=1}^{+\infty} \varepsilon_j u_1^j\right) \in \mathcal{E}_{p_1}(\Omega_1),$$

and

$$u_2 = \left(\sum_{j=1}^{+\infty} \varepsilon_j u_2^j\right) \in \mathcal{E}_{p_2}(\Omega_2).$$

Set $u = \max(u_1, u_2)$. Then Corollary 3.1 yields that $u \in \mathcal{E}_{p_1+p_2}(\Omega_1 \times \Omega_2)$. Furthermore, our construction implies that

$$u \leq \varepsilon_j \max(u_1^j, u_2^j) = \varepsilon_j u^j,$$

and

$$u^j \notin \mathcal{E}_{q_j}(\Omega).$$

Hence, $u \notin \mathcal{E}_{q_j}(\Omega_1 \times \Omega_2)$ for all $j \in \mathbb{N}$. For the argument of contradiction, assume that $u \notin \mathcal{E}_q(\Omega_1 \times \Omega_2)$ for some $q \neq p$. Without loss of generality assume that $q > p$. From Lemma 3.2 it now follows that $u \in \mathcal{E}_t(\Omega_1 \times \Omega_2)$ for all $p \leq t \leq q$. Fix $j_0 > 0$ such that $p < q_{j_0} < q$. Then $u \in \mathcal{E}_{q_{j_0}}$, and a contradiction is obtained, and this example is completed.

In [13] (see also [4]), Guedj and Zeriahi introduced the following formalism: For an increasing function $\chi\colon (-\infty, 0] \to (-\infty, 0]$, they say that a plurisubharmonic function u is in $\mathcal{E}_\chi(\Omega)$ if there exists a decreasing sequence $[u_j]$, $u_j \in \mathcal{E}_0$, that converges pointwise to u on Ω, as j tends to $+\infty$, and

$$\sup_{j\geq 1} \int_\Omega -\chi(u_j)(dd^c u_j)^n < +\infty.$$

For example, if $\chi(t) = -(-t)^p$, then $\mathcal{E}_\chi = \mathcal{E}_p$, and if χ is bounded with $\chi(0) \neq 0$, then $\mathcal{E}_\chi = \mathcal{F}$. In general, we do not have that $\mathcal{E}_\chi$ is contained in $\mathcal{E}$. Another consequence of our main is Corollary 3.4.

Corollary 3.4. *Assume that $\Omega_1 \subset \mathbb{C}^{n_1}$, $n_1 \geq 1$, and $\Omega_2 \subset \mathbb{C}^{n_2}$, $n_2 \geq 1$, are two bounded hyperconvex domains. Let $\chi_1, \chi_2\colon (-\infty, 0] \to (-\infty, 0]$ be increasing functions, $u_1 \in \mathcal{E}_{\chi_1}(\Omega_1)$, and $u_2 \in \mathcal{E}_{\chi_2}(\Omega_2)$. If $u(z_1, z_2) = \max(u_1(z_1), u_2(z_2))$, then $u \in \mathcal{E}_{-\chi_1\chi_2}(\Omega_1 \times \Omega_2)$.*

Proof. Let $\Omega = \Omega_1 \times \Omega_2$, $n = n_1 + n_2$, and let $[u_1^j]$, $[u_2^j]$ be sequences as in the proof of Corollary 2.1. Set $u^j = \max(u_1^j, u_2^j)$. From Corollary 2.2 with $h = \chi_1\chi_2$, and Fubini's theorem it follows that

$$\begin{aligned}
\varlimsup_{j\to\infty} \int_\Omega \chi_1(u^j)\chi_2(u^j)(dd^c u^j)^n &= \varlimsup_{j\to\infty} \int_\Omega \chi_1(u^j)\chi_2(u^j)(dd^c u_1^j)^{n_1} \wedge (dd^c u_2^j)^{n_2} \\
&\leq \varlimsup_{j\to\infty} \int_\Omega \chi_1(u_1^j)\chi_2(u_2^j)(dd^c u_1^j)^{n_1} \wedge (dd^c u_2^j)^{n_2} \\
&\leq \varlimsup_{j\to\infty} \int_{\Omega_1} \chi_1(u_1^j)(dd^c u_1^j)^{n_1} \int_{\Omega_2} \chi_2(u_2^j)(dd^c u_2^j)^{n_2} < +\infty.
\end{aligned}$$

Hence $u \in \mathcal{E}_{-\chi_1\chi_2}(\Omega_1 \times \Omega_2)$. □

4. The connection between $\max(u_1, u_2)$ and $(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}$

Proposition 4.1. *Assume that $\Omega \subset \mathbb{C}^n$, $n \geq 1$, is a bounded hyperconvex domain, and let $u_1, u_2 \in \mathcal{E}(\Omega)$. If $u = \max(u_1, u_2)$ and $(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}$ vanishes on pluripolar sets, then*

$$(dd^c u)^{n_1+n_2} \geq \chi_{\{u_1=u_2\}}(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}, \tag{4.1}$$

where $\chi_{\{u_1=u_2\}}$ *is the characteristic function for the set* $\{u_1 = u_2\}$ *in* Ω.

Proof. Without loss of generality we can assume that $u_1, u_2 < 0$. Let $[\alpha_j]$, $0 < \alpha_j < 1$, be an increasing sequence of real number that converges to 1, as $j \to +\infty$. By in [16] we have that

$$\begin{aligned}&(dd^c \max(\alpha_j u_1, u_2))^{n_1} \wedge (dd^c \max(u_1, \alpha_j u_2))^{n_2}\\ &\geq \chi_{\{\alpha_j u_1 > u_2\}\cap\{u_1<\alpha_j u_2\}}(dd^c\alpha_j u_1)^{n_1} \wedge (dd^c\alpha_j u_2)^{n_2}\\ &\geq \alpha_j^{n_1+n_2}\chi_{\{u_1=u_2\}}(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}.\end{aligned}$$

Let $j \to +\infty$, then (4.1) is obtained. □

Corollary 4.2. *Assume that* $\Omega \subset \mathbb{C}^n$, $n \geq 1$, *and let* $u_1, u_2 \in \mathcal{F}(\Omega)$ *be such that*

$$\int_{\{u_1\neq u_2\}} (dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2} = 0,$$

and $(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}$ *vanishes on pluripolar sets. If* $u = \max(u_1, u_2)$, *then* $(dd^c u)^{n_1+n_2} = (dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}$.

Proof. Note that

$$\int_\Omega (dd^c u)^{n_1+n_2} \leq \int_\Omega (dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}.$$ □

Corollary 4.3. *Assume that* $\Omega_1 \subset \mathbb{C}^{n_1}$, $n_1 \geq 1$, *and* $\Omega_2 \subset \mathbb{C}^{n_2}$, $n_2 \geq 1$, *are two bounded hyperconvex domains,* $u_1 \in \mathcal{F}(\Omega_1)$, $u_2 \in \mathcal{F}(\Omega_2)$, *and* $u_1, u_2 \in \mathcal{E}(\Omega_1 \times \Omega_2)$ *be such that* $(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}$ *vanishes on pluripolar sets. Set* $u(z_1, z_2) = \max(u_1(z_1), u_2(z_2))$. *Then* $(dd^c u)^{n_1+n_2} = (dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}$ *if, and only if,*

$$\int_{\{u_1\neq u_2\}} (dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2} = 0.$$

Proof. If $(dd^c u)^{n_1+n_2} = (dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}$, then we have $\int_{\{u_1\neq u_2\}}(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2} = 0$. On the other hand, we have by Proposition 4.1 that

$$(dd^c u)^n \geq \chi_{\{u_1=u_2\}}(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}$$

and, by Corollary 2.1, $\int(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2} = \int(dd^c u)^n$. Therefore, if

$$\int_{\{u_1\neq u_2\}} (dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2} = 0,$$

then it follows that $(dd^c u)^{n_1+n_2} = (dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}$. □

REMARK. The case when u_1 and u_2 are positive plurisubharmonic functions with

$$\int_{\{u_1>0\}} (dd^c u_1)^{n_1} = \int_{\{u_2>0\}} (dd^c u_2)^{n_2} = 0,$$

was proved in [5].

EXAMPLE 4.4. Let $u_1 = \max((1/2)\ln|z_1|, \ln|z_2|)$, and $u_2 = 2u_1$, then $(dd^c u_1)^n = (dd^c \max(u_1, u_2))^n = (1/2)\delta_0$. But $dd^c u_1 \wedge dd^c u_2 = \delta_0$. This shows that the condition: $(dd^c u_1)^{n_1} \wedge (dd^c u_2)^{n_2}$ vanishes on pluripolar sets, is necessary in Proposition 4.1.

Let $u \in \mathcal{E}$, then by Theorem 5.11 in [7] there exist functions $\phi_u \in \mathcal{E}_0$ and $f_u \in L^1_{\text{loc}}((dd^c \phi_u)^n)$, $f_u \geq 0$ such that $(dd^c u)^n = f_u (dd^c \phi_u)^n + \beta_u$. The non-negative measure β_u is such that there exists a pluripolar set $A \subseteq \Omega$ such that $\beta_u(\Omega \setminus A) = 0$. We shall use the notation that $\alpha_u = f_u (dd^c \phi_u)^n$ and β_u refereing to the decomposition discussed here.

Theorem 4.5. *Assume that* $\Omega_1 \subset \mathbb{C}^{n_1}$, $n_1 \geq 1$, *and* $\Omega_2 \subset \mathbb{C}^{n_2}$, $n_2 \geq 1$, *are two bounded hyperconvex domains, and let* $u_1 \in \mathcal{E}(\Omega_1)$, $u_2 \in \mathcal{E}(\Omega_2)$. *If* $u(z_1, z_2) = \max(u_1(z_1), u_2(z_2))$, *then*

$$\beta_u = \beta_{u_1} \otimes \beta_{u_2}.$$

Proof. Set $n = n_1 + n_2$. Assume first that if $\alpha_{u_j} = 0$, $j = 1, 2$. If we apply Corollary 4.3 to $\max(u_j, m)$, $j = 1, 2$ and let m tend to $-\infty$ we get that

$$(dd^c u)^n = (dd^c \max(u_1, u_2)))^{n_1+n_2} = (dd^c u_1)^{n_1} \otimes (dd^c u_2)^{n_2}. \tag{4.2}$$

For the general case we can without loss of generality assume that $u_1 \in \mathcal{F}(\Omega_1)$, $u_2 \in \mathcal{F}(\Omega_2)$. From [7] and Theorem 1 in [18] (or [1]), it follows that we can find functions such that for $j = 1, 2$ satisfies the following properties:

- $\varphi_j \in \mathcal{F}(\Omega_j)$, $v_j \in \mathcal{F}(\Omega_j)$,
- $(dd^c \varphi_j)^n$ vanishes on pluripolar sets,
- $(dd^c \varphi_j)^n = \alpha_{u_j}$, $(dd^c v_j)^n = \beta_{u_j}$,
- $\varphi_j \geq u_j$, $v_j \geq u_j$, and $u_j \geq \varphi_j + v_j$.

We now have that

$$\max(v_1, v_2) + \max(\varphi_1, v_2) + \max(v_1, \varphi_2) + \max(\varphi_1, \varphi_2) \leq \max(u_1, u_2) \leq \max(v_1, v_2).$$

By [7] every function $\varphi \in \mathcal{F}$ with $(dd^c \varphi)^n$ vanishing on all pluripolar sets can be minorized by the sum of a bounded function and a function with arbitrarily small Monge–Ampère mass. Using Corollary 2.1 we thus find that the following measures vanish on pluripolar sets:

$$(dd^c \max(\varphi_1, v_2))^{n_1+n_1}, \quad (dd^c \max(v_1, \varphi_2))^{n_1+n_2}, \quad (dd^c \max(\varphi_1, \varphi_2))^{n_1+n_2}.$$

Hence (4.2) and Lemma 4.11 in [1] concludes this proof since then

$$\beta_u = \beta_{\max(u_1,u_2)} = \beta_{\max(v_1,v_2)} = \beta_{v_1} \otimes \beta_{v_2} = \beta_{u_1} \otimes \beta_{u_2}. \qquad \square$$

EXAMPLE 4.6. If $\varphi \in \mathcal{PSH}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$, then

$$\int_K (-\psi)(dd^c\varphi)^n < +\infty \quad \text{for all} \quad K \Subset \Omega,\ \psi \in \mathcal{PSH}(\Omega),\ \psi \leq 0.$$

The following example shows that there exists a function $\varphi \in \mathcal{E}_0(\mathbb{D}^2)$, such that

$$\int_{\mathbb{D}^2} (-\ln|z_1|)(dd^c\varphi)^2 = +\infty.$$

Set

$$\varphi(z) = \sum_{j=1}^{+\infty} \max\left(\frac{\ln|z_1|}{j^6},\ j^2 \ln|z_2|, -\frac{1}{j^2}\right),$$

then by Corollary 4.3 we have that

$$\left(dd^c \max\left(\frac{\ln|z_1|}{j^6},\ j^2 \ln|z_2|, -\frac{1}{j^2}\right)\right)^2 = \frac{1}{j^4} d\sigma_{\{\ln|z_1|=-j^4\}} \otimes d\sigma_{\{\ln|z_2|x=-1/j^4\}}.$$

Lemma 2.5 in [9] implies that $\varphi \in \mathcal{E}_0(\mathbb{D}^2)$. Furthermore, it holds that

$$(dd^c\varphi)^2 \geq \sum_{j=1}^{+\infty} \frac{1}{j^4} d\sigma_{\{\ln|z_1|=-j^4\}} \otimes d\sigma_{\{\ln|z_2|=-1/j^4\}},$$

and therefore

$$\begin{aligned} &\int_{\mathbb{D}^2} (-\ln|z_1|)(dd^c\varphi)^2 \\ &\geq \sum_{j=1}^{+\infty} \frac{1}{j^4} \int_{\mathbb{D}^2} (-\ln|z_1|)\, d\sigma_{\{\ln|z_1|=-j^4\}} \otimes d\sigma_{\{\ln|z_2|=-1/j^4\}} = \sum_{j=1}^{+\infty} \frac{1}{j^4} j^4 = +\infty. \end{aligned}$$

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